

FEKETE-SZEGÖ PROBLEM FOR GENERALIZED BI-SUBORDINATE FUNCTIONS OF COMPLEX ORDER

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ABSTRACT. In this paper, we obtain Fekete-Szegő inequality for the generalized bi-subordinate functions of complex order. The results, which are presented in this study, would generalize those in related works of several earlier authors.

1. INTRODUCTION

Let \mathcal{A} be the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let \mathcal{S} be the class of functions f that are analytic and univalent in \mathbb{D} and are of the form

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

A function $f \in \mathcal{A}$ is said to be subordinate to a function $g \in \mathcal{A}$, denoted by $f \prec g$, if there exists a function $w \in \mathcal{B}_0$ where

$$\mathcal{B}_0 := \{w \in \mathcal{A} : w(0) = 0, |w(z)| < 1, (z \in \mathbb{D})\},$$

such that

$$f(z) = g(w(z)), \quad (z \in \mathbb{D}).$$

We let \mathcal{S}^* consist of starlike functions $f \in \mathcal{A}$, that is $\Re\{zf'(z)/f(z)\} > 0$ in \mathbb{D} and \mathcal{C} consist of convex functions $f \in \mathcal{A}$, that is $1 + \Re\{zf''(z)/f'(z)\} > 0$ in \mathbb{D} . In terms of subordination, these conditions are, respectively, equivalent to

$$\mathcal{S}^* \equiv \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \right\}$$

and

$$\mathcal{C} \equiv \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \right\}.$$

A generalization of the above two classes, according to Ma and Minda [23], are

$$\mathcal{S}^*(\varphi) \equiv \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{C}(\varphi) \equiv \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}$$

where φ is a positive real part function normalized by $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps D onto a region starlike with respect to 1 and symmetric with respect to the real axis. Obvious extensions of the above two classes (see [22]) are

$$\mathcal{S}^*(\gamma; \varphi) \equiv \left\{ f \in \mathcal{A} : 1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \varphi(z); \quad \gamma \in \mathbb{C} \setminus \{0\} \right\}$$

and

$$\mathcal{C}(\gamma; \varphi) \equiv \left\{ f \in \mathcal{A} : 1 + \frac{1}{\gamma} \left(\frac{zf''(z)}{f'(z)} \right) \prec \varphi(z); \quad \gamma \in \mathbb{C} \setminus \{0\} \right\}.$$

In literature, the functions belonging to these classes are called Ma-Minda *starlike and convex of complex order* γ ($\gamma \in \mathbb{C} \setminus \{0\}$), respectively.

Some of the special cases of the above two classes $\mathcal{S}^*(\gamma; \varphi)$ and $\mathcal{C}(\gamma; \varphi)$ are

Key words and phrases. Analytic functions, starlike functions, convex functions, Ma-Minda starlike functions, Ma-Minda convex functions, subordination, Fekete-Szegő Inequality.

2010 *Mathematics Subject Classification:* Primary 30C45; Secondary 30C80.

- (1) $\mathcal{S}^*((1, (1 + Az)/(1 + Bz))) = \mathcal{S}[A, B]$ and $\mathcal{C}(1, (1 + Az)/(1 + Bz)) = \mathcal{C}[A, B]$, $(-1 \leq B < A \leq 1)$ are classes of Janowski starlike and convex functions, respectively,
- (2) $\mathcal{S}^*((1 - \beta)e^{-i\delta} \cos \delta, (1 + z)/(1 - z)) = \mathcal{S}^*[\delta, \beta]$ and $\mathcal{C}((1 - \beta)e^{-i\delta} \cos \delta, (1 + z)/(1 - z)) = \mathcal{C}[\delta, \beta]$, $(|\delta| < \pi/2, 0 \leq \beta < 1)$ are classes of δ -spirallike and δ -Robertson univalent functions of order β , respectively,
- (3) $\mathcal{S}^*(1, (1 + (1 - 2\beta)z)/(1 - z)) = \mathcal{S}^*(\beta)$ and $\mathcal{C}(1, (1 + (1 - 2\beta)z)/(1 - z)) = \mathcal{C}(\beta)$ $(0 \leq \beta < 1)$ are classes of starlike and convex functions of order β , respectively,
- (4) $\mathcal{S}^*\left(1, \left(\frac{1+z}{1-z}\right)^\beta\right) = \mathcal{S}_\beta^*$ and $\mathcal{C}\left(1, \left(\frac{1+z}{1-z}\right)^\beta\right) = \mathcal{C}_\beta$ are class of strongly starlike and convex functions of order β , respectively,
- (5) $\mathcal{S}^*(1, \sqrt{1+z}) = \mathcal{S}_L^* = \left\{f \in \mathcal{A} : \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < 1\right\}$ is class of lemniscate starlike functions,
- (6) $\mathcal{S}^*(\gamma, (1+z)/(1-z)) = \mathcal{S}^*[\gamma]$ and $\mathcal{C}(\gamma, (1+z)/(1-z)) = \mathcal{C}[\gamma]$ $(\gamma \in \mathbb{C} \setminus \{0\})$ are classes of starlike and convex functions of complex order, respectively,
- (7) $\mathcal{S}^*(1, q_k(z)) = k - \mathcal{S}_P^* = \left\{f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > k \left|\frac{zf'(z)}{f(z)} - 1\right|\right\}$ is class of k -parabolik starlike functions,
- (8) $\mathcal{C}(1, q_k(z)) = k - \mathcal{UCV} = \left\{f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left|\frac{zf''(z)}{f'(z)}\right|\right\}$ is class of k -uniformly convex functions.

Here, for $0 \leq k < \infty$ the function $q_k : \mathbb{D} \rightarrow \{w = u + iv \in \mathbb{C} : u^2 > k^2((u-1)^2 + v^2), u > 0\}$ has the form $q_k(z) = 1 + Q_1 z + Q_2 z^2 + \dots, (z \in \mathbb{D})$ where

$$(1.2) \quad Q_1 = \begin{cases} \frac{2B^2}{1-k^2}; & 0 \leq k < 1, \\ \frac{8}{\pi^2}; & k = 1, \\ \frac{\pi^2}{4(k^2-1)\sqrt{t(1+t)}\mathcal{K}^2(t)}; & k > 1, \end{cases} \quad Q_2 = \begin{cases} \frac{(B^2+2)}{3}Q_1; & 0 \leq k < 1, \\ \frac{2}{3}Q_1; & k = 1, \\ \frac{4\mathcal{K}^2(t)(t^2+6t+1)-\pi^2}{24\sqrt{t(1+t)}\mathcal{K}^2(t)}Q_1; & k > 1, \end{cases}$$

with $B = \frac{2}{\pi} \arccos k$ and $\mathcal{K}(t)$ is the complete elliptic integral of first kind (see [20]).

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathbb{D} if both f and its inverse map f^{-1} are univalent in \mathbb{D} . Let σ be the class functions $f \in \mathcal{S}$ that are bi-univalent in \mathbb{D} . For a brief history and interesting examples of functions which are in (or are not in) in the class σ , including various properties of such functions we refer the reader to the work of Srivastava et al. [5] and references therein. Bounds for the first few coefficients of various subclasses of bi-univalent functions were obtained by a variety of authors including [19, 13, 2], [1], [21], [6, 7, 17, 18]. Not much was known about the bounds of the general coefficients $a_n; n \geq 4$ of subclasses of σ up until the publication of the article [11] by Jahangiri and Hamidi and followed by a number of related publications. Moreover, many author have considered the Fekete-Szegő problem for various subclasses of \mathcal{A} , the upper bound for $|a_3 - \mu a_2^2|$ is investigated by many different authors (see [[16], [15], [9], [8]]). In this paper, we apply the Fekete-Szegő inequality to certain subclass of generalized bi-subordinate functions of complex order.

2. Coefficient Estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk \mathbb{D} , satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function is known to be typically real with the series expansion $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ where B_1, B_2 are real and $B_1 > 0$. Motivated by a class of functions defined by the first author [2], we define the following comprehensive class of analytic functions

$$\mathcal{S}(\lambda, \gamma; \varphi) \equiv \left\{f \in \mathcal{A} : 1 + \frac{1}{\gamma} \left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) - \lambda z f'(z)} - 1 \right) \prec \varphi(z); 0 \leq \lambda \leq 1, \gamma \in \mathbb{C} \setminus \{0\} \right\}.$$

A function $f \in \mathcal{A}$ is said to be generalized bi-subordinate of complex order γ and type λ if both f and its inverse map $g = f^{-1}$ are in $\mathcal{S}(\lambda, \gamma; \varphi)$. As special cases of the class $\mathcal{S}(\lambda, \gamma; \varphi)$ we have $\mathcal{S}(0, \gamma; \varphi) \equiv \mathcal{S}^*(\gamma; \varphi)$ and $\mathcal{S}(1, \gamma; \varphi) \equiv \mathcal{C}(\gamma; \varphi)$.

To prove our next theorems, we shall need the following well-known lemma (see [3]).

Lemma 2.1. (see [3]) Let the function $w \in \mathcal{B}_0$ be given by

$$w(z) = c_1 z + c_2 z^2 + \dots (z \in \mathbb{D}),$$

then for by every complex number s ,

$$|c_2 - sc_1^2| \leq 1 + (|s| - 1) |c_1|^2.$$

In the following theorem, we consider fuctional $|a_3 - \mu a_2^2|$ for γ nonzero complex number and $\mu \in \mathbb{C}$.

Theorem 2.1. *Let γ be a nonzero complex number, $\mu \in \mathbb{C}$ and $0 \leq \lambda \leq 1$. If both functions f of the form (1.1) and its inverse maps $g = f^{-1}$ are in $\mathcal{S}(\lambda, \gamma; \varphi)$, then we obtain,*

$$(2.1) \quad |a_2| \leq \frac{|\gamma| B_1}{(1 + \lambda)},$$

$$(2.2) \quad |a_3| \leq \frac{|\gamma| |B_1|}{4(1 + 2\lambda)} \max\{2, (|s| + |t|)\}$$

and

$$(2.3) \quad |a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1 |\gamma|}{2(1+2\lambda)} & \text{if } \mathcal{L} \leq 2, \\ \frac{B_1 |\gamma|}{4(1+2\lambda)} L & \text{if } \mathcal{L} \geq 2, \end{cases}$$

where $s = \frac{B_2}{B_1} - \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}$, $t = \frac{B_2}{B_1}$ and $\mathcal{L} = \left| \frac{B_2}{B_1} + (1 - \mu) \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2} \right| + \left| \frac{B_2}{B_1} \right|$.

Proof. Let $f(z) \in \mathcal{S}(\lambda, \gamma; \varphi)$ and $g = f^{-1}$. Then there are two functions $u(z) = c_1 z + c_2 z^2 + \dots$ ($z \in \mathbb{D}$) and $v(w) = d_1 w + d_2 w^2 + \dots$, such that

$$(2.4) \quad \begin{aligned} & 1 + \frac{1}{\gamma} \left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) - \lambda z f'(z)} - 1 \right) = \varphi(u(z)) \\ & = 1 + \frac{(1 + \lambda) a_2}{\gamma} z + \left[\frac{2(1 + 2\lambda) a_3 - (1 - \lambda)^2 a_2^2}{\gamma} \right] z^2 + \dots \end{aligned}$$

and

$$(2.5) \quad \begin{aligned} & 1 + \frac{1}{\gamma} \left(\frac{wg'(w) + \lambda w^2 g''(w)}{(1 - \lambda)g(w) - \lambda w g'(w)} - 1 \right) = \varphi(v(w)) \\ & = 1 - \frac{(1 + \lambda) a_2}{\gamma} w + \left[\frac{-2(1 + 2\lambda) a_3 + (3 + 6\lambda - \lambda^2) a_2^2}{\gamma} \right] w^2 + \dots \end{aligned}$$

Equating coefficients of both side of equations (2.4) and (2.5) yield

$$(2.6) \quad \frac{(1 + \lambda) a_2}{\gamma} = B_1 c_1, \quad \frac{2(1 + 2\lambda) a_3 - (1 - \lambda)^2 a_2^2}{\gamma} = B_1 c_2 + B_2 c_1^2,$$

$$(2.7) \quad \frac{-(1 + \lambda) a_2}{\gamma} = B_1 d_1, \quad \frac{-2(1 + 2\lambda) a_3 + (3 + 6\lambda - \lambda^2) a_2^2}{\gamma} = B_1 d_2 + B_2 d_1^2,$$

so that, on account of (2.6) and (2.7)

$$(2.8) \quad c_1 = -d_1,$$

$$(2.9) \quad a_2 = \frac{\gamma B_1}{(1 + \lambda)} c_1$$

and

$$(2.10) \quad a_3 = a_2^2 + \frac{\gamma}{4(1 + 2\lambda)} [B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2], \quad |c_k| \leq 1.$$

Taking into account (2.8), (2.9), (2.10) and Lemma 2.1, we obtain

$$(2.11) \quad |a_2| = \left| \frac{\gamma B_1}{(1 + \lambda)} c_1 \right| \leq \frac{|\gamma| B_1}{(1 + \lambda)}$$

and

$$\begin{aligned}
|a_3| &= \left| a_2^2 + \frac{\gamma}{4(1+2\lambda)} [B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2] \right| \\
&= \left| \frac{\gamma^2 B_1^2}{(1+\lambda)^2} c_1^2 + \frac{\gamma}{4(1+2\lambda)} [(B_1 c_2 - B_2 d_1^2) - (B_1 d_2 - B_2 c_1^2)] \right| \\
&= \left| \frac{\gamma^2 B_1^2}{(1+\lambda)^2} c_1^2 + \frac{\gamma}{4(1+2\lambda)} [(B_1 c_2 - B_2 c_1^2) - (B_1 d_2 - B_2 d_1^2)] \right| \\
&= \left| \frac{\gamma B_1}{4(1+2\lambda)} \left\{ c_2 - \left(\frac{B_2}{B_1} - \frac{4\gamma B_1(1+2\lambda)}{(1+\lambda)^2} \right) c_1^2 \right\} - \left[d_2 - \frac{B_2}{B_1} d_1^2 \right] \right| \\
&\leq \frac{|\gamma| B_1}{4(1+2\lambda)} \left\{ \left| c_2 - \left(\frac{B_2}{B_1} - \frac{4\gamma B_1(1+2\lambda)}{(1+\lambda)^2} \right) c_1^2 \right| + \left| d_2 - \frac{B_2}{B_1} d_1^2 \right| \right\} \\
&\leq \frac{|\gamma| B_1}{4(1+2\lambda)} \{ 1 + (|s| - 1) |c_1^2| + 1 + (|t| - 1) |c_1^2| \} \\
&= \frac{|\gamma| B_1}{4(1+2\lambda)} \{ 2 + (|s| + |t| - 2) |c_1^2| \} \\
&\leq \frac{|\gamma| B_1}{4(1+2\lambda)} \max \{ 2, (|s| + |t|) \}.
\end{aligned}$$

Thus, we have

$$|a_3| \leq \frac{|\gamma| |B_1|}{4(1+2\lambda)} \max \{ 2, (|s| + |t|) \},$$

where $s = \frac{B_2}{B_1} - \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2}$ and $t = \frac{B_2}{B_1}$.

Furthermore,

$$\begin{aligned}
|a_3 - \mu a_2^2| &= \left| (1 - \mu) a_2^2 + \frac{\gamma}{4(1+2\lambda)} [B_1 c_2 + B_2 c_1^2 - B_1 d_2 - B_2 d_1^2] \right| \\
&= \left| \frac{\gamma B_1}{4(1+2\lambda)} \left\{ \left[c_2 - \left(\frac{B_2}{B_1} - (1 - \mu) \frac{4\gamma B_1(1+2\lambda)}{(1+\lambda)^2} \right) c_1^2 \right] - \left[d_2 - \frac{B_2}{B_1} d_1^2 \right] \right\} \right| \\
(2.12) \quad &\leq \frac{|\gamma| B_1}{4(1+2\lambda)} \left\{ 2 + \left(\left| \frac{B_2}{B_1} - (1 - \mu) \frac{4\gamma B_1(1+2\lambda)}{(1+\lambda)^2} \right| + \left| \frac{B_2}{B_1} \right| - 2 \right) |c_1^2| \right\}.
\end{aligned}$$

As a result of this, we obtain

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1 |\gamma|}{2(1+2\lambda)} & \text{if } \mathcal{L} < 2, \\ \frac{B_1 |\gamma|}{4(1+2\lambda)} \mathcal{L} & \text{if } \mathcal{L} \geq 2, \end{cases}.$$

where $\mathcal{L} = \left| \frac{B_2}{B_1} + (1 - \mu) \frac{4B_1\gamma(1+2\lambda)}{(1+\lambda)^2} \right| + \left| \frac{B_2}{B_1} \right|$.

Thus the proof is completed. \square

We next consider the case, when γ and μ are real. Then we have:

Theorem 2.2. *Let $\gamma > 0$ and if both functions f of the form (1.1) and its inverse maps $g = f^{-1}$ are in $\mathcal{S}(\lambda, \gamma; \varphi)$, then for $\mu \in \mathbb{R}$,*

(1) *If $|B_2| \geq B_1$, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\gamma |B_2|}{2(1+2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} & \text{if } \mu \leq 1, \\ \frac{\gamma |B_2|}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} & \text{if } \mu > 1, \end{cases}$$

(2) *If $|B_2| < B_1$, we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\gamma |B_2|}{2(1+2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} & \text{if } \mu \leq 1 - \mathcal{F}, \\ \frac{\gamma B_1}{2(1+2\lambda)} & \text{if } 1 - \mathcal{F} < \mu < 1 + \mathcal{F}, \\ \frac{\gamma |B_2|}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} & \text{if } \mu \geq 1 + \mathcal{F}, \end{cases}$$

where $\mathcal{F} = \frac{(1+\lambda)^2(B_1 - |B_2|)}{2\gamma B_1^2(1+2\lambda)}$.

For each μ there is a function $f \in \mathcal{S}(\lambda, \gamma; \varphi)$ such that equality holds.

Proof. Using (2.12) and Lemma 2.1, we obtain

$$\begin{aligned}
 |a_3 - \mu a_2^2| &= \left| \frac{\gamma B_1}{4(1+2\lambda)} \left\{ \left[c_2 - \left(\frac{B_2}{B_1} - (1-\mu) \frac{4\gamma B_1(1+2\lambda)}{(1+\lambda)^2} \right) c_1^2 \right] - \left[d_2 - \frac{B_2}{B_1} d_1^2 \right] \right\} \right| \\
 &\leq \frac{|\gamma| B_1}{4(1+2\lambda)} \left\{ 2 + \left(\left| \frac{B_2}{B_1} - (1-\mu) \frac{4\gamma B_1(1+2\lambda)}{(1+\lambda)^2} \right| + \left| \frac{B_2}{B_1} \right| - 2 \right) |c_1^2| \right\} \\
 (2.13) \quad &= \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + |\mu - 1| \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} |c_1^2|.
 \end{aligned}$$

Now, the proof will be presented in two cases by considering $B_1, B_2 \in \mathbb{R}$ and $B_1 > 0$.

Firstly, we want to consider the case with $|B_2| \geq B_1$. Several possible cases need to further analyze.

Case 1 : If $\mu \leq 1$, using (2.13) and Lemma 2.1, we obtain

$$\begin{aligned}
 |a_3 - \mu a_2^2| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} |c_1^2| \\
 &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} \\
 &= \frac{\gamma B_1}{2(1+2\lambda)} + \frac{\gamma|B_2|}{2(1+2\lambda)} - \frac{\gamma B_1}{2(1+2\lambda)} + \frac{\gamma^2 B_1^2}{(1+\lambda)^2} - \mu \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \\
 &= \frac{\gamma|B_2|}{2(1+2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2}.
 \end{aligned}$$

Case 2 : If $\mu > 1$, again using (2.13) and Lemma 2.1, we obtain

$$\begin{aligned}
 |a_3 - \mu a_2^2| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} |c_1^2| \\
 &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} \\
 &= \frac{\gamma B_1}{2(1+2\lambda)} + \frac{\gamma|B_2|}{2(1+2\lambda)} - \frac{\gamma B_1}{2(1+2\lambda)} - \frac{\gamma^2 B_1^2}{(1+\lambda)^2} + \mu \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \\
 &= \frac{\gamma|B_2|}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2}.
 \end{aligned}$$

Finally, we want to consider the case with $|B_2| < B_1$. By a similar way, several possible cases need to further analyze.

(i) Let $\mu \leq 1 - \mathcal{F}$, using (2.13) and Lemma 2.1, we have

$$\begin{aligned}
 |a_3 - \mu a_2^2| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} |c_1^2| \\
 &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} \\
 &= \frac{\gamma|B_2|}{2(1+2\lambda)} - (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2}.
 \end{aligned}$$

(ii) Let $1 - \mathcal{F} < \mu \leq 1$, using (2.13) and Lemma 2.1, we obtain

$$\begin{aligned}
 |a_3 - \mu a_2^2| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (1-\mu) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} |c_1^2| \\
 &\leq \frac{\gamma B_1}{2(1+2\lambda)}.
 \end{aligned}$$

(iii) Let $1 < \mu < 1 + \mathcal{F}$, using (2.13) and Lemma 2.1, we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} |c_1^2| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)}. \end{aligned}$$

(iv) Let $\mu \geq 1 + \mathcal{F}$, using (2.13) and Lemma 2.1, we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} |c_1^2| \\ &\leq \frac{\gamma B_1}{2(1+2\lambda)} + \left\{ \frac{\gamma(|B_2| - B_1)}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \right\} \\ &= \frac{\gamma B_1}{2(1+2\lambda)} + \frac{\gamma|B_2|}{2(1+2\lambda)} - \frac{\gamma B_1}{2(1+2\lambda)} - \frac{\gamma^2 B_1^2}{(1+\lambda)^2} + \mu \frac{\gamma^2 B_1^2}{(1+\lambda)^2} \\ &= \frac{\gamma|B_2|}{2(1+2\lambda)} + (\mu - 1) \frac{\gamma^2 B_1^2}{(1+\lambda)^2}. \end{aligned}$$

Thus the proof is completed. \square

Finally, we consider the case, when γ nonzero complex number and $\mu \in \mathbb{C}$. Then we get:

Theorem 2.3. Let γ be a nonzero complex number and let both functions f of the form (1.1) and its inverse maps $g = f^{-1}$ are in $\mathcal{S}(\lambda, \gamma; \varphi)$. Then for $\mu \in \mathbb{R}$ we have

(1) If $\frac{(1+|\sin \theta|)|B_2|}{2B_1} \geq 1$, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin \theta|)}{4(1+2\lambda)} & \text{if } \mu \leq 1 - \Re(k_1), \\ \frac{|\gamma||B_2|(1+|\sin \theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) & \text{if } \mu > 1 - \Re(k_1). \end{cases}$$

(2) If $\frac{(1+|\sin \theta|)|B_2|}{2B_1} < 1$, we obtain

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) + \frac{|\gamma||B_2|(1+|\sin \theta|)}{4(1+2\lambda)} & \text{if } \mu \leq 1 - \Re(k_1) + \mathcal{N}, \\ \frac{|\gamma|B_1}{2(1+2\lambda)} & \text{if } 1 - \Re(k_1) + \mathcal{N} < \mu < 1 - \Re(k_1) - \mathcal{N}, \\ \frac{|\gamma||B_2|(1+|\sin \theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} (1 - \mu - \Re(k_1)) & \text{if } \mu \geq 1 - \Re(k_1) - \mathcal{N}, \end{cases}$$

$$\text{where } k_1 = \frac{B_2(1+\lambda)^2 e^{i\theta}}{4B_1^2|\gamma|(1+2\lambda)}, l_1 = \frac{(|B_2| - 2B_1)(1+\lambda)^2}{4B_1^2|\gamma|(1+2\lambda)}, |\gamma| = \gamma e^{i\theta} \text{ and } \mathcal{N} = \frac{(1+\lambda)^2[|B_2|(1+|\sin \theta|) - 2B_1]}{4B_1^2|\gamma|(1+2\lambda)}.$$

For each μ there is a function in $\mathcal{S}(\lambda, \gamma; \varphi)$ such that the equality holds.

Proof. Suppose $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{S}(\lambda, \gamma; \varphi)$, using (2.12) and Lemma 2.1, then we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{|\gamma| B_1}{4(1+2\lambda)} \left\{ 2 + \left(\left| \frac{B_2}{B_1} - (1 - \mu) \frac{4\gamma B_1(1+2\lambda)}{(1+\lambda)^2} \right| + \left| \frac{B_2}{B_1} \right| - 2 \right) |c_1^2| \right\} \\ (2.14) \quad &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \left[\left| (1 - \mu) - \frac{B_2(1+\lambda)^2}{4B_1^2\gamma(1+2\lambda)} \right| + \frac{(|B_2| - 2B_1)(1+\lambda)^2}{4B_1^2|\gamma|(1+2\lambda)} \right] |c_1^2|. \end{aligned}$$

Taking $|\gamma| = \gamma e^{i\theta}$, $k_1 = \frac{B_2(1+\lambda)^2 e^{i\theta}}{4B_1^2|\gamma|(1+2\lambda)}$ and $l_1 = \frac{(|B_2|-2B_1)(1+\lambda)^2}{4B_1^2|\gamma|(1+2\lambda)}$, a direct calculation with (2.14) shows that

$$\begin{aligned}
 |a_3 - \mu a_2^2| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} (|1 - \mu - k_1| + l_1) |c_1^2| \\
 &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} (|1 - \mu - \Re(k_1) - i(\mathbf{Im}(k_1))| + l_1) |c_1^2| \\
 &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \left(|1 - \mu - \Re(k_1)| + \frac{|B_2|(1+\lambda)^2 |\sin \theta|}{4B_1^2 |\gamma|(1+2\lambda)} + l_1 \right) |c_1^2| \\
 &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \left(|1 - \mu - \Re(k_1)| + \frac{|B_2| |\sin \theta|}{|B_2| - 2B_1} l_1 + l_1 \right) |c_1^2| \\
 &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \left[|1 - \mu - \Re(k_1)| + l_1 \left(\frac{|B_2| |\sin \theta|}{|B_2| - 2B_1} + 1 \right) \right] |c_1^2| \\
 (2.15) \quad &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma| [|B_2|(1+|\sin \theta|) - 2B_1]}{4(1+2\lambda)} \right] |c_1^2|.
 \end{aligned}$$

Now, we will make some discussions for several different cases by considering $B_1, B_2 \in \mathbb{R}$ and $B_1 > 0$.

Firstly, we want to consider the case with $\frac{2B_1}{|B_2|} - |\sin \theta| < 1$. Several possible cases need to further analyze.

Case 1 : Let $\mu \leq 1 - \Re(k_1)$. Then from (2.15) and Lemma 2.1, we obtain

$$\begin{aligned}
 |a_3 - \mu a_2^2| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma| [|B_2|(1+|\sin \theta|) - 2B_1]}{4(1+2\lambda)} \right] |c_1^2| \\
 &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma| [|B_2|(1+|\sin \theta|) - 2B_1]}{4(1+2\lambda)} \\
 &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma| |B_2|(1+|\sin \theta|)}{4(1+2\lambda)} - \frac{|\gamma| B_1}{2(1+2\lambda)} \\
 &= \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma| |B_2|(1+|\sin \theta|)}{4(1+2\lambda)}.
 \end{aligned}$$

Case 2 : Let $\mu > 1 - \Re(k_1)$, then from (2.15) and Lemma 2.1, we yield

$$\begin{aligned}
 |a_3 - \mu a_2^2| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma| [|B_2|(1+|\sin \theta|) - 2B_1]}{4(1+2\lambda)} \right] |c_1^2| \\
 &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (\mu + \Re(k_1) - 1) + \frac{|\gamma| [|B_2|(1+|\sin \theta|) - 2B_1]}{4(1+2\lambda)} \\
 &= \frac{|\gamma| B_1}{2(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma| |B_2|(1+|\sin \theta|)}{4(1+2\lambda)} - \frac{|\gamma| B_1}{2(1+2\lambda)} \\
 &= \frac{|\gamma| |B_2|(1+|\sin \theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)).
 \end{aligned}$$

Finally, we want to consider the case with $\frac{2B_1}{|B_2|} - |\sin \theta| > 1$. By a similar approximation, several possible cases need to further analyze.

(i) Let $\mu \leq 1 - \Re(k_1) + \mathcal{N}$, using (2.15) and Lemma 2.1, we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma| [|B_2| (1 + |\sin \theta|) - 2B_1]}{4(1+2\lambda)} \right] |c_1^2| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma| [|B_2| (1 + |\sin \theta|) - 2B_1]}{4(1+2\lambda)} \\ &= \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma| |B_2| (1 + |\sin \theta|)}{4(1+2\lambda)} - \frac{|\gamma| B_1}{2(1+2\lambda)} \\ &= \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma| |B_2| (1 + |\sin \theta|)}{4(1+2\lambda)}. \end{aligned}$$

(ii) Let $1 - \Re(k_1) + \mathcal{N} < \mu \leq 1 - \Re(k_1)$, using (2.15) and Lemma 2.1, we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma| [|B_2| (1 + |\sin \theta|) - 2B_1]}{4(1+2\lambda)} \right] |c_1^2| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)}. \end{aligned}$$

(iii) Let $1 - \Re(k_1) < \mu < 1 - \Re(k_1) - \mathcal{N}$, using (2.15) and Lemma 2.1, we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma| [|B_2| (1 + |\sin \theta|) - 2B_1]}{4(1+2\lambda)} \right] |c_1^2| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)}. \end{aligned}$$

(iv) Let $\mu \geq 1 - \Re(k_1) - \mathcal{N}$, using (2.15) and Lemma 2.1, we have

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \left[\frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot |1 - \mu - \Re(k_1)| + \frac{|\gamma| [|B_2| (1 + |\sin \theta|) - 2B_1]}{4(1+2\lambda)} \right] |c_1^2| \\ &\leq \frac{|\gamma| B_1}{2(1+2\lambda)} + \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (\mu + \Re(k_1) - 1) + \frac{|\gamma| [|B_2| (1 + |\sin \theta|) - 2B_1]}{4(1+2\lambda)} \\ &= \frac{|\gamma| B_1}{2(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)) + \frac{|\gamma| |B_2| (1 + |\sin \theta|)}{4(1+2\lambda)} - \frac{|\gamma| B_1}{2(1+2\lambda)} \\ &= \frac{|\gamma| |B_2| (1 + |\sin \theta|)}{4(1+2\lambda)} - \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2} \cdot (1 - \mu - \Re(k_1)). \end{aligned}$$

Thus the proof is completed. \square

Corollary 2.4. Let $\gamma = 1$ and $\lambda = 0$. If both functions f of the form (1.1) and its inverse maps $g = f^{-1}$ are in $\mathcal{S}[A, B]$, then using Theorem (2.1), (2.2) and (2.3), we obtain

(1) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{A-B}{2} & \text{if } |B| + |4(1-\mu)(A-B) - B| < 2, \\ \frac{A-B}{4} [|B| + |4(1-\mu)(A-B) - B|] & \text{if } |B| + |4(1-\mu)(A-B) - B| \geq 2. \end{cases}$$

(2) For $\gamma > 0$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|B|(A-B)}{2} - (\mu - 1)(A-B)^2 & \text{if } \mu \leq 1 - \frac{1-|B|}{2(A-B)}, \\ \frac{A-B}{2} & \text{if } 1 - \frac{1-|B|}{2(A-B)} < \mu < 1 + \frac{1-|B|}{2(A-B)}, \\ \frac{|B|(A-B)}{2} + (\mu - 1)(A-B)^2 & \text{if } \mu \geq 1 + \frac{1-|B|}{2(A-B)}, \end{cases}$$

(3) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} (A-B)^2 (1-\mu) + \frac{|B|(A-B)(1+|\sin \theta| - \cos \theta)}{4} & \text{if } \mu \leq 1 + \psi_1(A, B, \theta), \\ \frac{A-B}{2} & \text{if } 1 + \psi_1(A, B, \theta) < \mu < 1 - \psi_2(A, B, \theta), \\ \frac{|B|(A-B)(1+|\sin \theta| + \cos \theta)}{4} - (A-B)^2 (1-\mu) & \text{if } \mu \geq 1 - \psi_2(A, B, \theta), \end{cases}$$

where $B_1 = A - B$, $B_2 = -B(A - B)$, $-1 \leq B < A \leq 1$, $\psi_1(A, B, \theta) = \frac{|B|(1+|\sin \theta|-\cos \theta)-2}{4(A-B)}$ and $\psi_2(A, B, \theta) = \frac{|B|(1+|\sin \theta|+\cos \theta)-2}{4(A-B)}$.

Corollary 2.5. Let $\gamma = 1$ and $\lambda = 1$. If both functions f of the form (1.1) and its inverse maps $g = f^{-1}$ are in $\mathcal{C}[A, B]$, then using Theorem (2.1), (2.2) and (2.3), we have

(1) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{A-B}{6} & \text{if } |B| + |3(1-\mu)(A-B) - B| < 2, \\ \frac{A-B}{12} [|B| + |3(1-\mu)(A-B) - B|] & \text{if } |B| + |3(1-\mu)(A-B) - B| \geq 2. \end{cases}$$

(2) For $\gamma > 0$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|B|(A-B)}{6} - (\mu-1) \frac{(A-B)^2}{4} & \text{if } \mu \leq 1 - \frac{2(1-|B|)}{3(A-B)}, \\ \frac{A-B}{6} & \text{if } 1 - \frac{2(1-|B|)}{3(A-B)} < \mu < 1 + \frac{2(1-|B|)}{3(A-B)}, \\ \frac{|B|(A-B)}{6} + (\mu-1) \frac{(A-B)^2}{4} & \text{if } \mu \geq 1 + \frac{2(1-|B|)}{3(A-B)}. \end{cases}$$

(3) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} (1-\mu) \frac{(A-B)^2}{4} + \frac{|B|(A-B)(1+|\sin \theta|-\frac{4}{3}\cos \theta)}{12} & \text{if } \mu \leq 1 + \phi_1(A, B, \theta), \\ \frac{A-B}{6} & \text{if } 1 + \phi_1(A, B, \theta) < \mu < 1 - \phi_2(A, B, \theta), \\ \frac{|B|(A-B)(1+|\sin \theta|+\frac{4}{3}\cos \theta)}{12} - (1-\mu) \frac{(A-B)^2}{4} & \text{if } \mu \geq 1 - \phi_2(A, B, \theta), \end{cases}$$

where $B_1 = A - B$, $B_2 = -B(A - B)$, $-1 \leq B < A \leq 1$, $\phi_1(A, B, \theta) = \frac{|B|(1+|\sin \theta|-\cos \theta)-2}{3(A-B)}$ and $\phi_2(A, B, \theta) = \frac{|B|(1+|\sin \theta|+\cos \theta)-2}{3(A-B)}$.

Corollary 2.6. Let $\gamma \in \mathbb{C} \setminus \{0\}$ and $\lambda = 0$. If both functions f of the form (1.1) and its inverse maps $g = f^{-1}$ are in $\mathcal{S}^*[\gamma]$, then similarly, using Theorem (2.1), (2.2) and (2.3), we obtain (i) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} |\gamma| & \text{if } |1 + (1-\mu)8\gamma| < 1, \\ \frac{|\gamma|}{2} [|1 + (1-\mu)8\gamma| + 1] & \text{if } |1 + (1-\mu)8\gamma| \geq 1. \end{cases}$$

(ii) For $\gamma > 0$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \gamma - 4(\mu-1)\gamma^2 & \text{if } \mu \leq 1, \\ \gamma + 4(\mu-1)\gamma^2 & \text{if } \mu > 1, \end{cases}$$

(iii) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4|\gamma|^2(1-\mu) + \frac{|\gamma|(1+|\sin \theta|-\cos \theta)}{2} & \text{if } \mu \leq 1 + \psi_1(\gamma, \theta), \\ |\gamma| & \text{if } 1 + \psi_1(\gamma, \theta) < \mu < 1 - \psi_2(\gamma, \theta), \\ \frac{|\gamma|(1+|\sin \theta|-\cos \theta)}{2} - 4|\gamma|^2(1-\mu) & \text{if } \mu \geq 1 - \psi_2(\gamma, \theta), \end{cases}$$

where $B_1 = 2$, $B_2 = 2$, $\psi_1(\gamma, \theta) = \frac{(|\sin \theta|-\cos \theta-1)}{8|\gamma|}$ and $\psi_2(\gamma, \theta) = \frac{(|\sin \theta|+\cos \theta-1)}{8|\gamma|}$.

Corollary 2.7. Let $\gamma \in \mathbb{C} \setminus \{0\}$ and $\lambda = 1$. Let both functions f of the form (1.1) and its inverse maps $g = f^{-1}$ are in $\mathcal{C}[\gamma]$. Then similarly, using Theorem (2.1), (2.2) and (2.3), we have

(i) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{|\gamma|}{3} & \text{if } |1 + (1-\mu)6\gamma| < 1, \\ \frac{|\gamma|}{2} [|1 + (1-\mu)6\gamma| + 1] & \text{if } |1 + (1-\mu)6\gamma| \geq 1. \end{cases}$$

(ii) For $\gamma > 0$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\gamma}{3} - (\mu-1)\gamma^2 & \text{if } \mu \leq 1, \\ \frac{\gamma}{3} + (\mu-1)\gamma^2 & \text{if } \mu > 1, \end{cases}$$

(iii) For $\gamma \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} |\gamma|^2(1-\mu) + \frac{|\gamma|(1+|\sin \theta|-\cos \theta)}{6} & \text{if } \mu \leq 1 + \phi_1(\gamma, \theta), \\ \frac{|\gamma|}{3} & \text{if } 1 + \phi_1(\gamma, \theta) < \mu < 1 - \phi_2(\gamma, \theta), \\ \frac{|\gamma|(1+|\sin \theta|-\cos \theta)}{6} - |\gamma|^2(1-\mu) & \text{if } \mu \geq 1 - \phi_2(\gamma, \theta), \end{cases}$$

where $B_1 = 2$, $B_2 = 2$, $\phi_1(\gamma, \theta) = \frac{(|\sin \theta| - \cos \theta - 1)}{6|\gamma|}$ and $\phi_2(\gamma, \theta) = \frac{(|\sin \theta| + \cos \theta - 1)}{6|\gamma|}$.

Acknowledgement 1. *The research of E. Deniz and M. Çağlar were supported by the Commission for the Scientific Research Projects of Kafkas University, project number 2016-FM-67.*

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